Price Discovery in Waiting Lists: A Connection to Stochastic Gradient Descent

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Price Discovery in Waiting Lists

Waiting times serve as prices in waiting lists

- Agents choose among items and associated waiting times
- Can be similar to standard competitive equilibria

Waiting list mechanisms are commonly used

- e.g., public housing, organ allocation,…

Natural price discovery process

- Planner does not set prices
- Prices determined by endogenous queue lengths
- Prices adjust with each arrival
  - Similar to Tâtonnement – price increases with demand (agents join queue), decreases with supply (items arrive)

Example – Queueing for One Item

- Single item, arrives at Poisson rate 1
- Agents arrive at Poisson rate 2
  - Agents observe the queue length, can join the queue or leave
  - Quasilinear utility

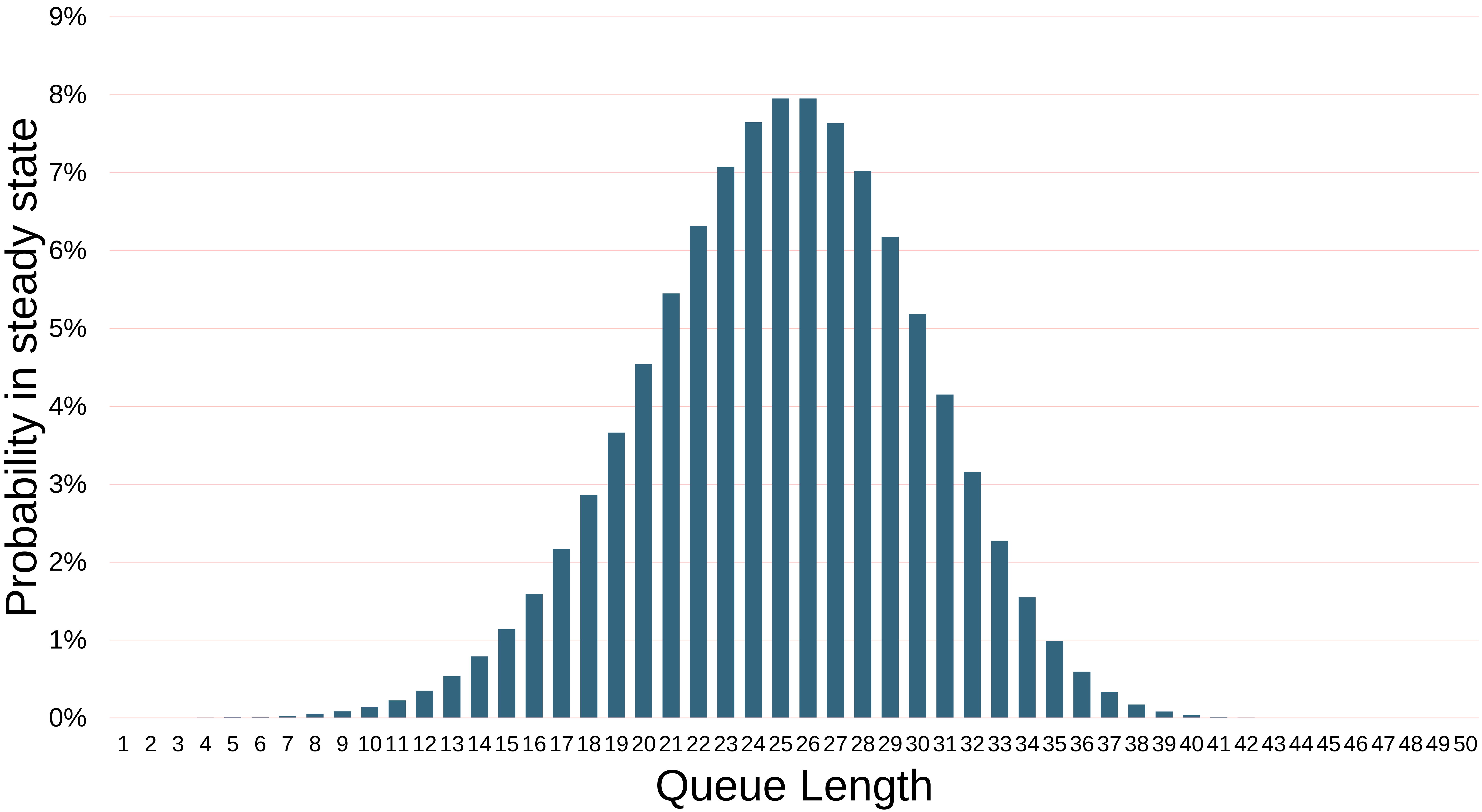
$$v - 0.02 \cdot w$$

with $v \sim U[0,1]$ i.i.d.

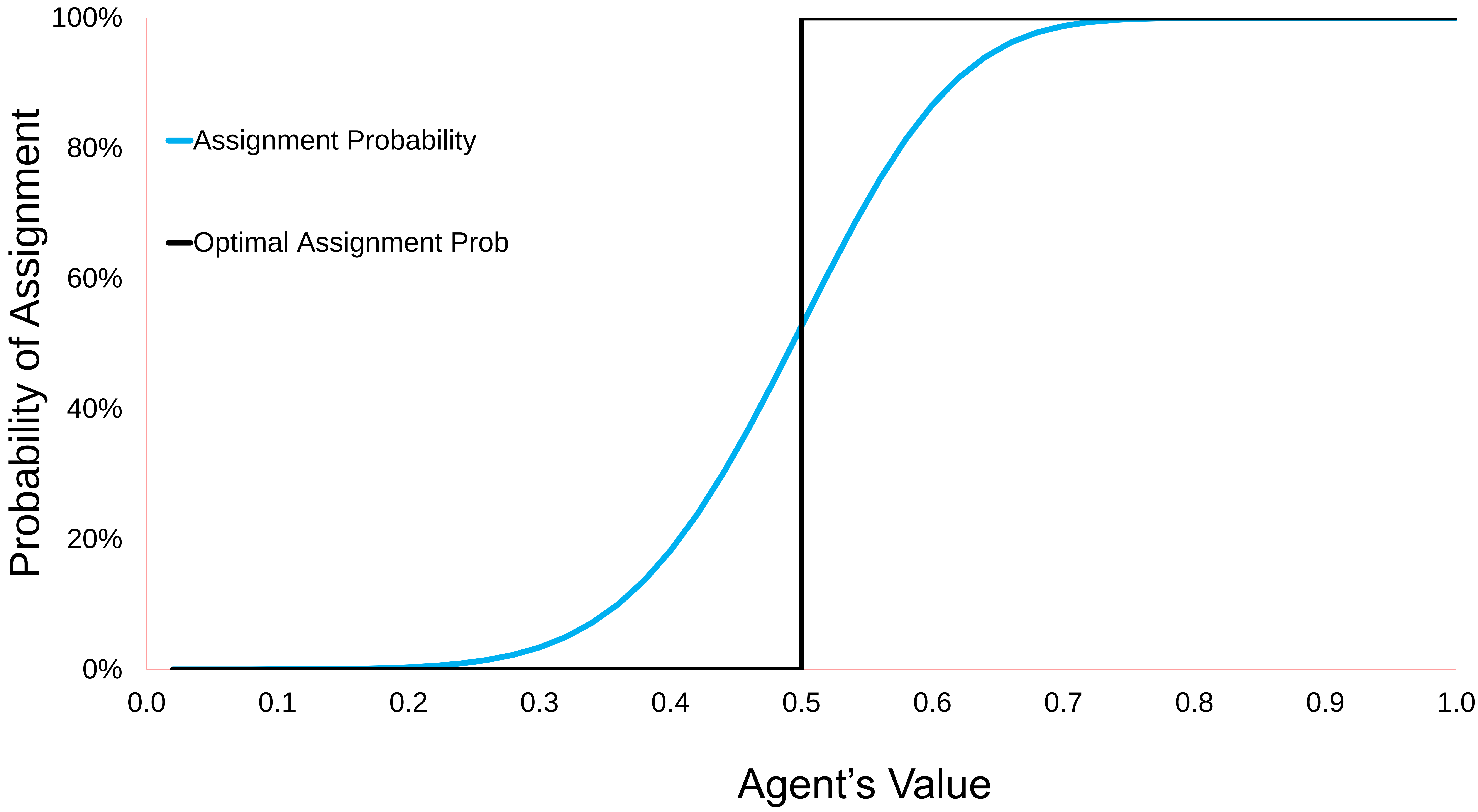
Static benchmark:

- Collect all items and agents that arrive until (large) time T
- Assigning agents if $v \geq 1/2$ maximizes allocative efficiency
- Market clearing price is $\mathrm{p}^* = 1/2$

Example – One Item



Example – One Item



Price Discovery in Waiting Lists

Question: Allocative efficiency under fluctuating prices

Main Result: Loss from price fluctuations is bounded by the *adjustment size*

- Bound is (almost) tight
- Conditions for when the loss is negligible

Methodological contribution:

- Price adaptation as a stochastic gradient decent (SGD)
- Duality, Lyapunov functions

Price rigidity: tradeoff between learning speed and overreaction

Related Work

Dynamic matching mechanisms:

- Leshno (2017), Baccara Lee and Yariv (2018), Bloch and Cantala (2017), Su and Zenios (2004), Arnosti and Shi (2017), Loertscher Muir Taylor (2020).

Convergence of tâtonnement processes using gradient descent:

- Cheung Cole and Devanur (2019), Cheung Cole and Tao (2018), Cole and Fleischer,(2008), Uzawa (1960).
- Correa and Stier-Moses (2010), Powell and Sheffi (1982).

Cost of fluctuations:

- Asker Collard-Wexler and De Loecker (2014), De Vany (1976), Carlton (1977), and Carlton (1978).

Model

Items: Arrive according to Poisson process, total rate $\mu = 1$

- Finite number of items $J_{\emptyset} = \{1,2,\dots,J\} \cup \{\emptyset\}$
- With probability μ_j arriving item is of type j

Agents: Arrive according to Poisson process with total rate λ

- Agent type $\theta \in \Theta$, drawn i.i.d. according to distribution F
- Possibly uncountably many or finitely many types

Quasi-Linear Utility:

- $\mathrm{u}_\theta(j,w)$ is the utility of type θ agent assigned item j with wait w

$$\mathrm{u}_\theta(j,w) = v(\theta,j) - c(w)$$

- Agents can leave immediately (balk) to obtain utility $v(\theta,\emptyset) = 0$
- Values are private information
- $v(\theta,j)$ is bounded; $c(\cdot)$ is smooth, strictly increasing and convex or concave

Assignments and Allocative Efficiency

Assignments $\boldsymbol{\eta}$

Let $\eta_t \in J_\emptyset$ denote the item assigned to agent who arrived at t

Allocative efficiency

$$W(\eta) = \liminf_{T\to\infty} \frac{1}{|\mathcal{A}_T|} \sum_{t\in\mathcal{A}_T} v(\theta_t, \eta_t)$$

Optimal allocative efficiency

$$W^{OPT} = \mathbb{E}\left[\sup_{\eta} W(\eta)\right]$$

- Restricting attention to assignments η that satisfy a no-Ponzi condition

The Waiting List Mechanism

Separate queue for each item $j \in J$

- First Come First Served (FCFS) assignment policy
- Agents who join a queue wait until assigned (no reneging)

Choice of agent θ who observes $\boldsymbol{q}$:

$$a(\theta, \mathbf{q}) = \underset{j \in \mathcal{J} \cup \{\emptyset\}}{\operatorname{argmax}} \left\{ v(\theta, j) - \mathbb{E}[c(w_j) | \mathbf{q}] \right\}$$

- Observes all queue lengths $\boldsymbol{q} = (q_1, .., q_J)$
- Can join any queue, or leave unassigned

- Simplified version of public housing assignment

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Choice of agent θ who observes $\boldsymbol{q}$:

$$a(\theta, \mathbf{q}) = \underset{j \in \mathcal{J} \cup \{\emptyset\}}{\operatorname{argmax}} \left\{ v(\theta, j) - p_j(\mathbf{q}) \right\}$$

- Observes state-dependent prices:

$$p_j(\boldsymbol{q}) = p_j(q_j) = \mathbb{E}[c(w_j) | q_j]$$

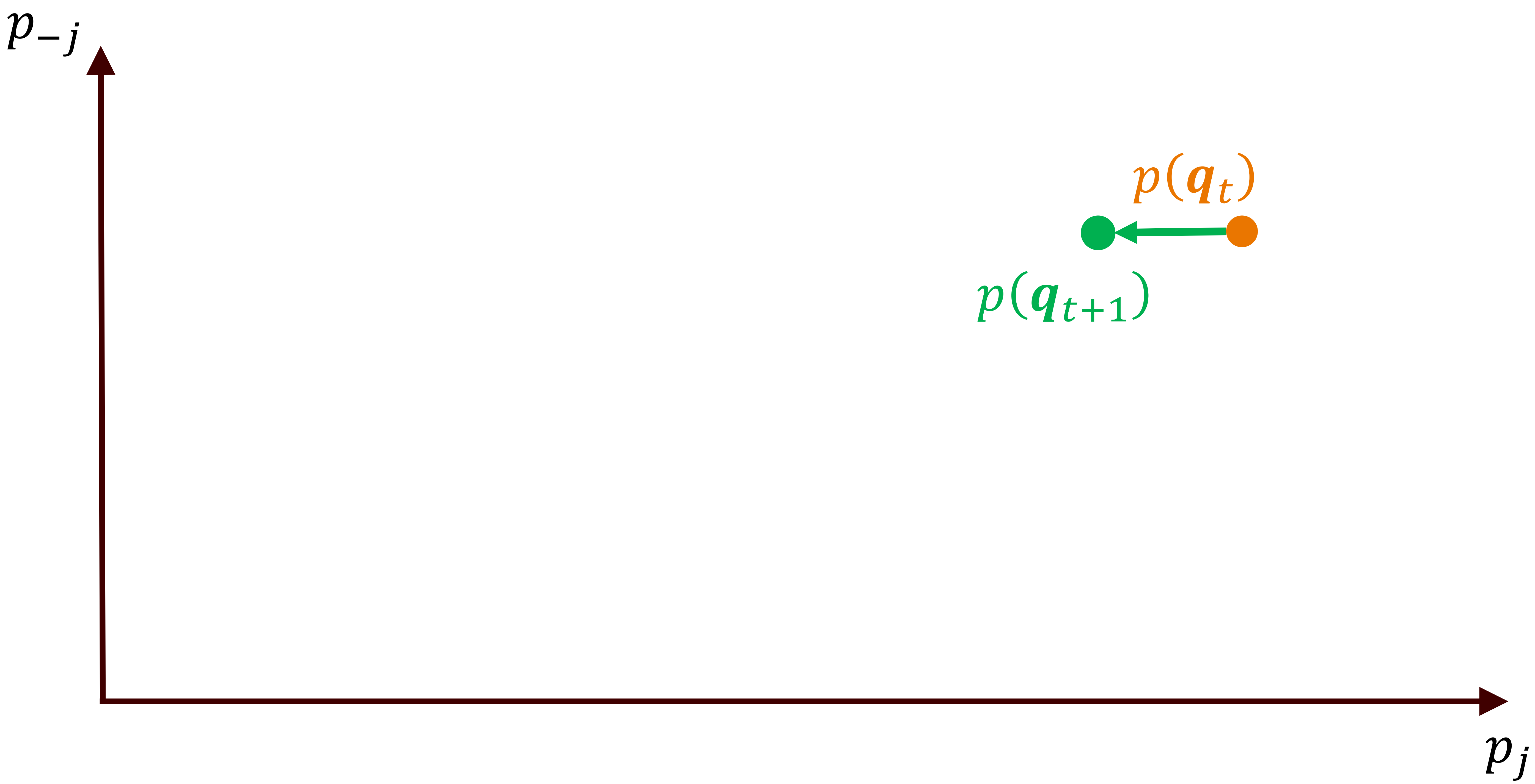
- Simplified version of public housing assignment

Stochastic Price Adaptation

Transition if agent arrives, sees queue lengths $\boldsymbol{q}_t$, joins queue j

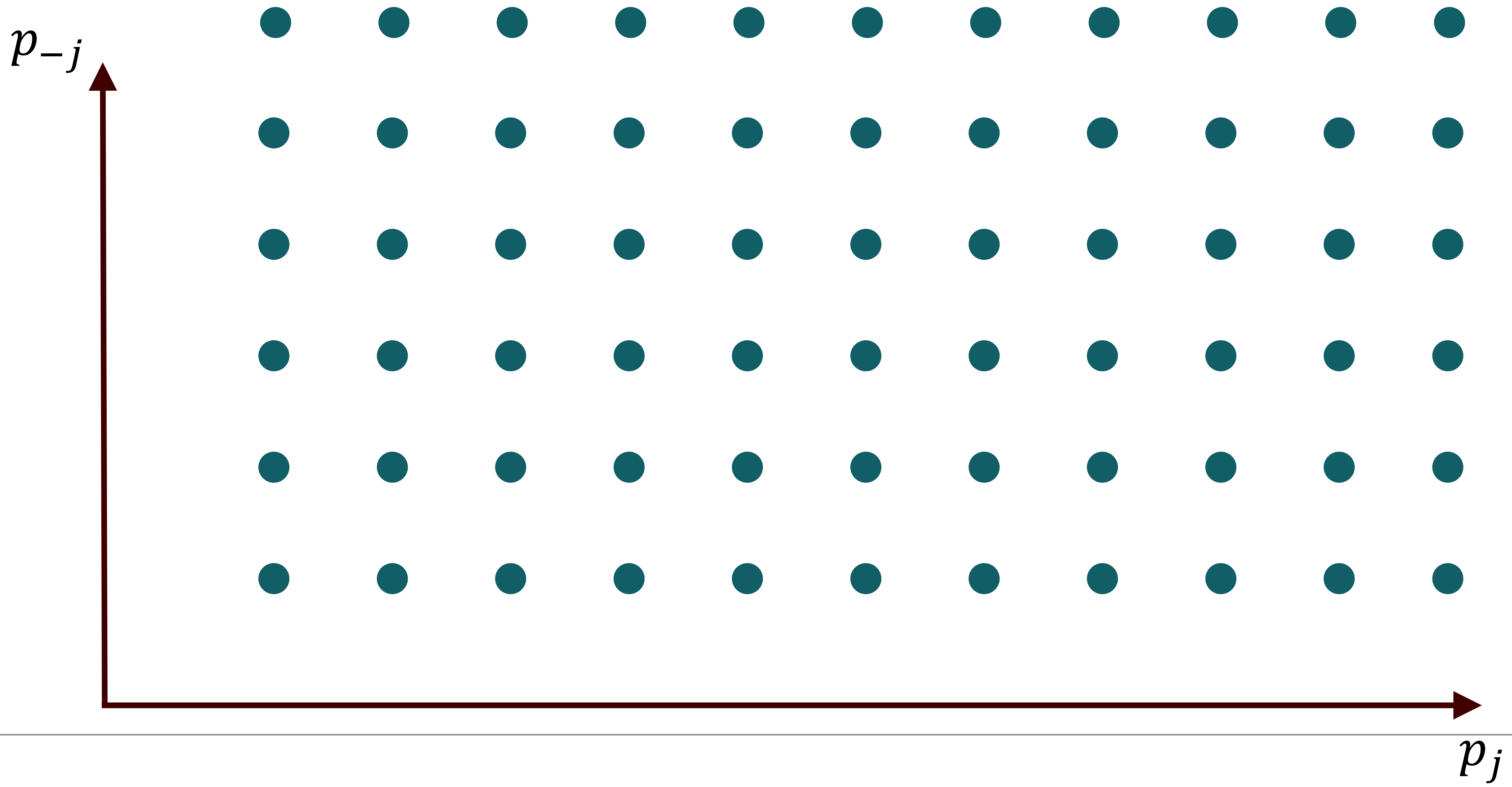
Stochastic Price Adaptation

Transition if item j arrives, assigned to an agent in queue j



Stochastic Price Adaptation

- Allocative efficiency W^{WL} is the expected match value under the steady state distribution
- When there are >2 items, the steady state distribution is not tractable



The Waiting List Mechanism

- The expected allocative efficiency under the waiting list is

$$W^{WL} = \mathbb{E}\left[W\left(\eta^{WL}\right)\right]$$

- Adjustment size Δ is defined by

$$\Delta = \max_{j \in \mathcal{J}} \max_{1 \leq q \leq q_{\max}} \{p_j(q) - p_j(q-1)\}$$

- If waiting costs are linear $c(w) = c \cdot w$, then

$$\Delta = {}^{c}/_{\mu_{min}}$$

is the cost of waiting for one item arrival.

Main Result: Bounding Allocative Efficiency

Theorem 1:

Allocative efficiency under the waiting list is bounded by

$$W^{WL} \geq W^{OPT} - \frac{\lambda + 2}{2\lambda} \Delta$$

Main Result: Bounding Allocative Efficiency

Theorem 1':

Suppose $p^* > 0$ for any market clearing p^*; $c(\cdot)$ is linear.

Then, allocative efficiency under the waiting list is

$$W^{WL} \geq W^{OPT} - \Delta - \varepsilon$$

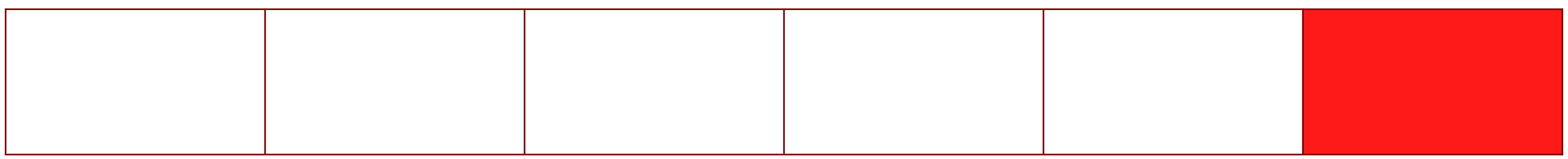
The allocative efficiency loss is bounded by the cost of waiting for one item arrival

- High loss if an apartment arrives monthly, low loss if apartments arrive daily

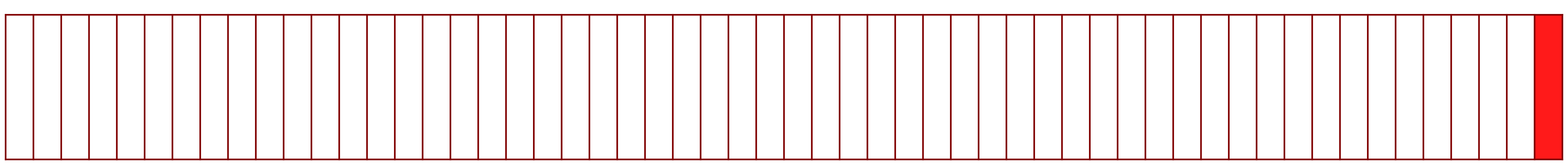
Main Result: Intuition

Suppose $p^* =$ cost of waiting six months

- If apartments arrives monthly, corresponding queue length is 5
- Each arrival significantly changes the price



- If apartments arrive daily, corresponding queue length is 180
- Each arrival slightly changes the price



Relation to Static Assignment

Let W^* be the optimal allocative efficiency in the corresponding static assignment problem:

$$\begin{aligned} W^* = \max_{\{x_{\theta j}\}_{\theta\in\Theta, j\in\mathcal{J}}} \quad & \sum_{j\in\mathcal{J}} \int_\Theta x_{\theta j}\, v(\theta, j)\, dF(\theta) \\ \text{subject to} \quad & \sum_{j\in\mathcal{J}} x_{\theta j} \le 1,\ x_{\theta j} \in [0,1] && \forall \theta \in \Theta \\ & \int_\Theta \lambda x_{\theta j}\, dF(\theta) \le \mu_j && \forall j \in \mathcal{J} \end{aligned}$$

Proposition:

$$W^{OPT} = W^*$$

Duality for the Static Assignment

Lemma (Monge-Kantorovich duality):

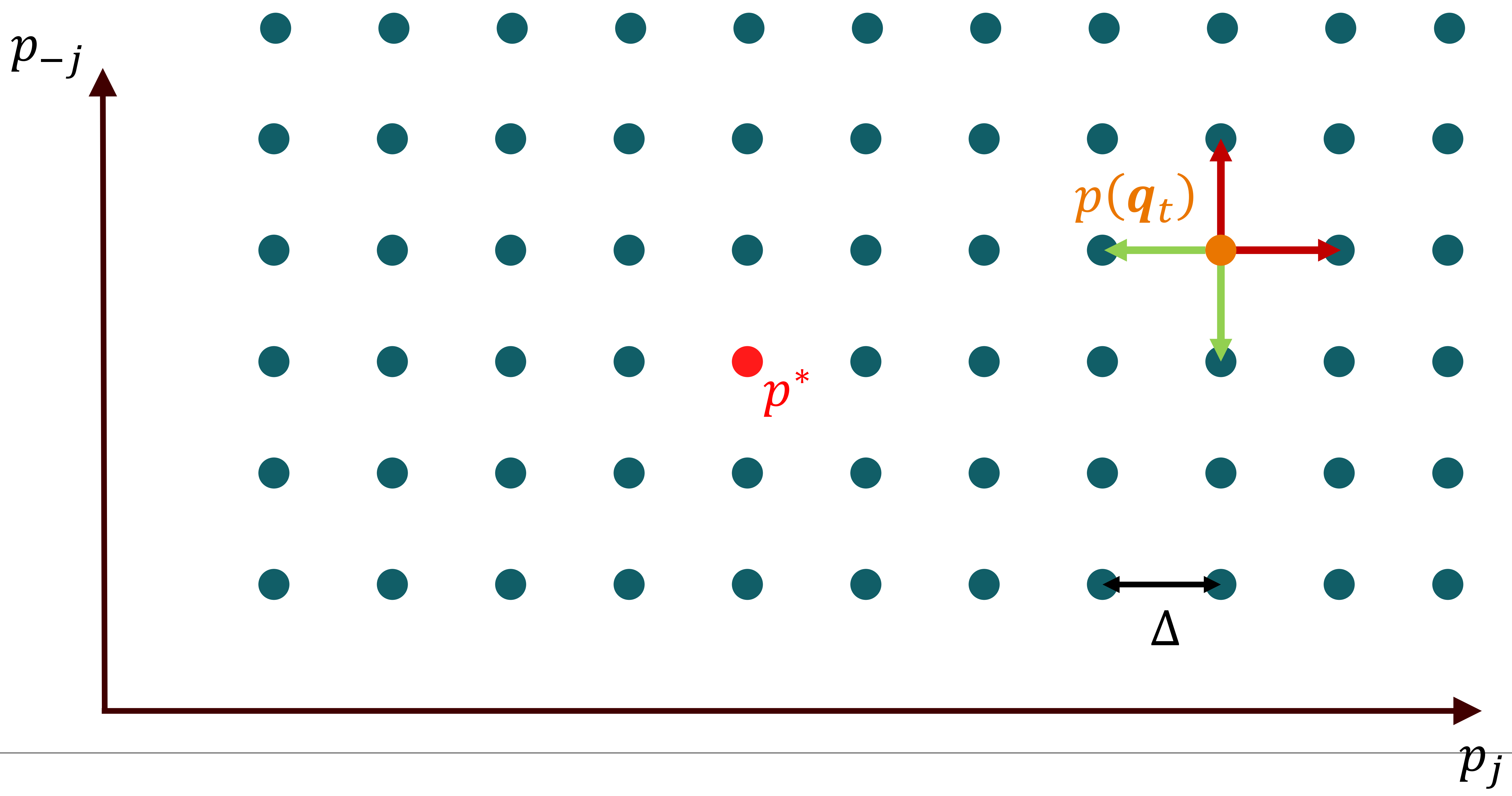
$$\min_{\boldsymbol{p} \geq \boldsymbol{0}} h(\boldsymbol{p}) = W^*$$

for

$$h(p) = \int_{\Theta} \max_{j \in J \cup \{\emptyset\}} \left[v(\theta, j) - p_j \right] + \frac{1}{\lambda} \sum_{j \in J} \mu_j p_j$$

Relation to Stochastic Gradient Descent

- Let p^* denote optimal static prices
- Prices $p(\boldsymbol{q}_t)$ change when an item arrives, or agent arrives
- Δ is the maximal adjustment size



Relation to Stochastic Gradient Descent

The expected adjustment is

$$\mathbb{E}[q_{j,t+1} - q_{j,t}] = \frac{\lambda}{1+\lambda} \int_{\Theta} \mathbf{1}_{\{a(\theta,\boldsymbol{q}_t)=j\}} dF(\theta) - \frac{1}{1+\lambda} \mu_j$$

which is a sub-gradient of the dual objective

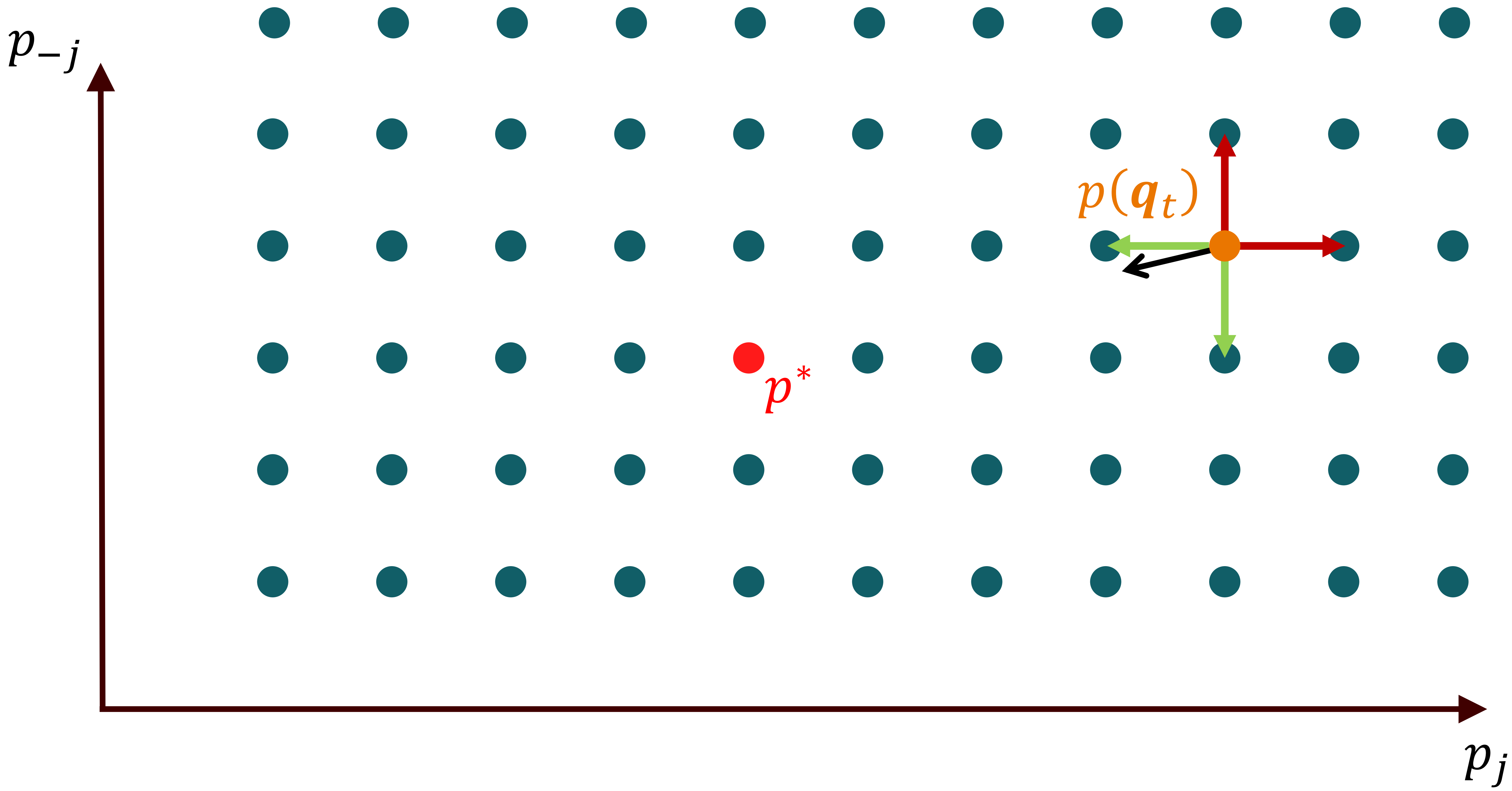
$$h(\boldsymbol{p}) = \int_{\Theta} \max_{j \in \mathcal{J} \cup \{\emptyset\}} \left[v(\theta, j) - p_j \right] dF(\theta) + \frac{1}{\lambda} \sum_{j \in \mathcal{J}} \mu_j p_j$$

That is, the expected step is in direction of a gradient decent

- Works for deep learning
- Unlike when SGD is used for optimization, step size Δ is fixed and does not shrink to 0

Relation to Stochastic Gradient Descent

- Prices moves towards p^* in expectation



Proof Sketch

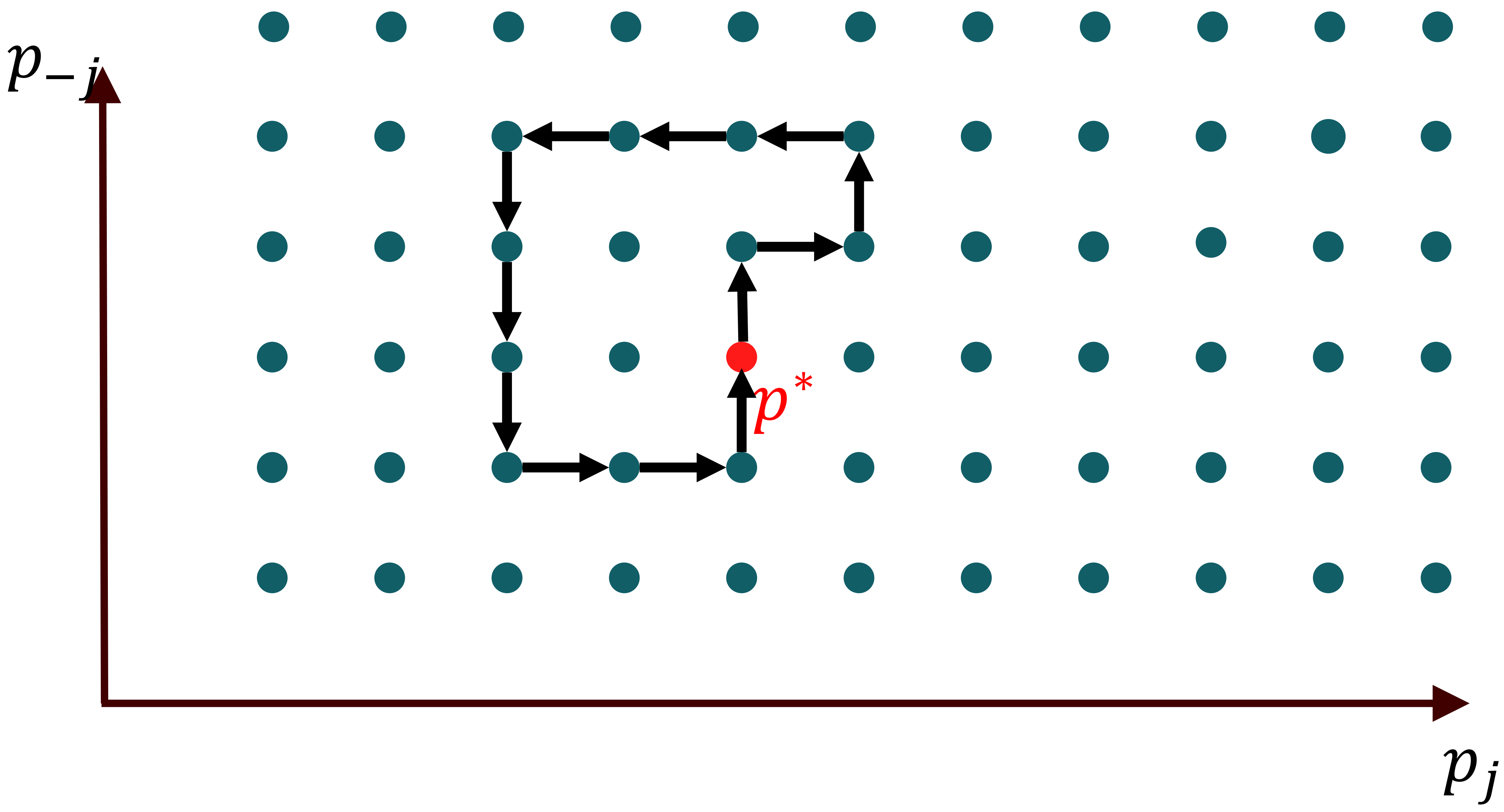
- Define a Lyapunov function $L(q)$ such that $\nabla L(q) = p(q)$
- Decompose the value generated from each arrival:

$$\mathbb{E}[v(\theta_t, a(\theta_t, \mathbf{q}_t))|\mathbf{q}_t] \geq \frac{\lambda}{\lambda+1} W^* - \underbrace{L(\mathbf{q}_t) - \mathbb{E}[L(\mathbf{q}_{t+1})|\mathbf{q}_t]}_{\text{(I) Change in Potential}} - \underbrace{\frac{2+\lambda}{2(1+\lambda)}\Delta}_{\text{(II) loss}}$$

Proof Sketch

- Over many periods, the potential term cancels out

$$\frac{1}{\mathrm{T}}\sum_{t=t_0}^{T}[L(\boldsymbol{q}_t)-L(\boldsymbol{q}_{t+1})]=\frac{1}{T}\Big(L\big(\boldsymbol{q}_{t_0}\big)-L(\boldsymbol{q}_T)\Big)\approx 0$$



Proof Sketch

- Decompose the value generated from each arrival:

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- After canceling (I), the loss per period is bounded by (II)
  - Bound is independent of $\boldsymbol{q}_t$, implying we do not need to calculate the stationary distribution

When is the Loss High?

Proposition 2:
For any number of items J there exist an economy where allocative efficiency is

$$W^{WL} \approx W^{OPT} - \Delta$$

Example of High Loss

- Agents $\Theta = J$, each agent only wants the corresponding item

$$v(\theta, j) = \mathbf{1}_{\{\theta = j\}}$$

- Identical arrival rates of items and corresponding agents

- Loss when an agent arrives and price is too high (maximal queue length)
- Loss proportional to $\Delta = c/\mu_j$
  - Queue lengths follow an unbiased reflected random walk
  - Queue lengths $q_j = 0,1,2, \dots, 1/\Delta$ equally likely in steady state
  - Probability of hitting the boundary is roughly ${}^{1}/_{1/\Delta}$.

When is the Loss Low?

Theorem 3:

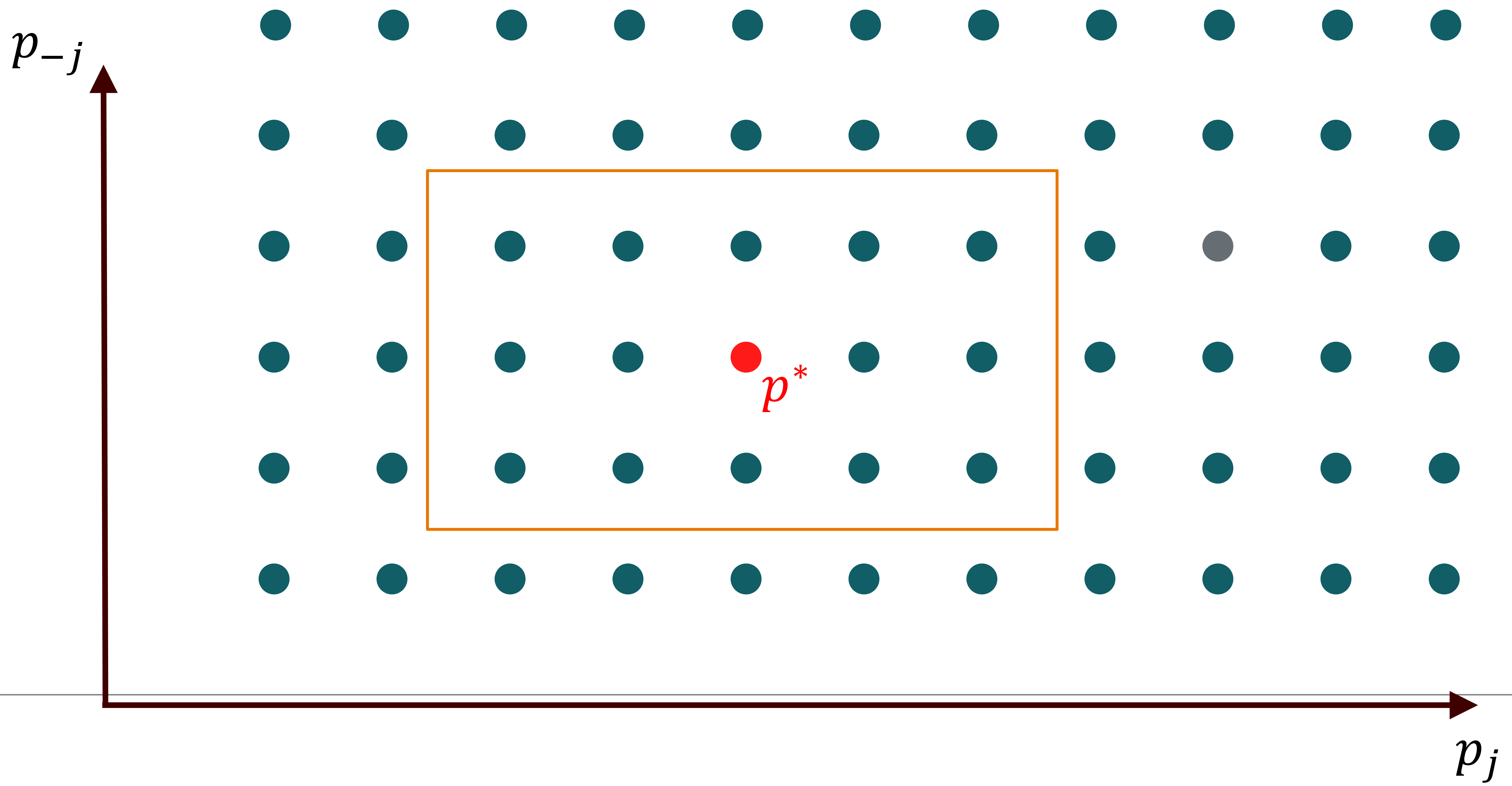
Consider an economy with finitely many agent types and linear waiting costs $c(w) = c \cdot w$. Suppose there is a unique market clearing price. Then there exist $\alpha, \beta, c_0 > 0$ such that for any $c < c_0$

$$W^{WL} \geq W^{OPT} - \beta e^{-\alpha/\Delta}$$

- Note: an economy with finitely many agents generically has a unique market clearing

Theorem 3: Stronger Concentration

- If the dual is unique, no loss within a neighborhood of p^*
  - Agents only take items they are assigned under the optimal assignment with positive probability
- Biased random walk towards p^*



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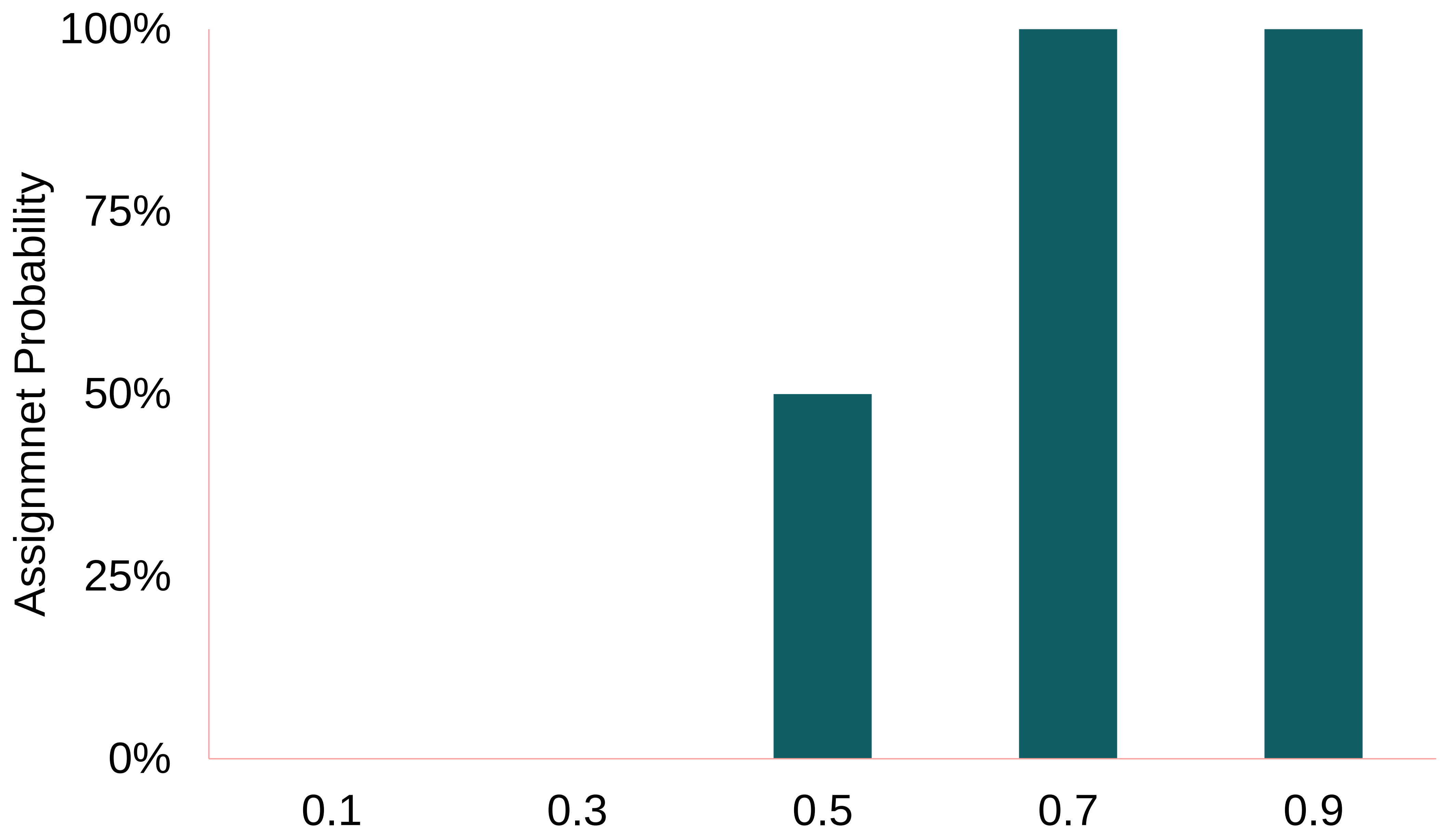

Optimal Adjustment Size and Price Rigidity

Consider a planner who can set prices,

but does not know the distribution of agent preferences

- Agents arrive over time, can learn from choices of past agents
- Finite horizon T

A simple pricing SGD pricing heuristic:

- Increase price of item j by Δ when an agent chooses j
- Decrease the price of item j by Δ at rate proportional to supply

Optimal Adjustment Size and Price Rigidity

Theorem:

The allocative efficiency of SGD pricing with adjustment size $\Delta = 1/\sqrt{T}$ is at least

$$W_T^{WL} \geq W_T^* - O\left(\sqrt{T}\right)$$

- Choice of intermediate Δ balances two sources of loss:
  - Smaller Δ implies less loss from price fluctuations
  - Larger Δ implies less transient loss during initial learning
- $O\left(\sqrt{T}\right)$ is the minimal possible loss (Devanur et al. 2019)

Optimal Adjustment Size and Price Rigidity

Attractive simple pricing heuristic

- Efficiency guarantees
- Algorithm can operate continuously, even if demand changes
- No knowledge required, apart from frequency of changes

Naturally occurring pricing rigidity

- Prices continuously adjust, unaware of changes in demand
  - e.g., do Fed announcements affect demand for Italian food?
- Slow reaction when demand does change
  - Algorithm unsure whether it observes new demand patters or noise
- No need for menu costs, rational inattention, etc.

Conclusion

- Analysis of allocative efficiency in waiting lists
  - Simple, natural price adaptation process
- Connection to stochastic gradient decent
  - Bounds through Lyapunov functions
- Random fluctuations cause an efficiency loss
  - Simple price adaptation policy can do well
  - Loss depends on the “adjustment size” – how much one arrival changes prices
- Pricing heuristic generates slow response to demand changes